On the classical roots of the Schroedinger equation

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Abstract

In semiclassical approximation Schroedinger equation is known to reduce to classical Hamilton-Jacobi equation. These equations look strikingly similar. An idea that just the Hamilton-Jacobi equation became a prototype of the Schroedinger equation arises. Arguments in favour of this assumption are supplied. Then it is no wonder that Aharonov-Bohm effect was recently derived directly from the classical Hamilton-Jacobi equation (without using Schroedinger equation), and hence, it is, in fact, of classical origin. The electron-field interaction is explained within the framework of classical electrodynamics. Thus, the so-called unlocal interaction becomes unnecessary.

Keywords: Schroedinger equation, Hamilton-Jacobi equation, Aharonov-Bohm effect

1 Introduction

Speaking on Quantum Mechanics we will mean Erwin Shroedinger and Werner Heisenberg mechanics created in 1925-1926 rather that Quantum Physics or Theory founded by Max Planck in 1901. An analogy between classical and quantum mechanics is known to exist. Some physical effects, for instance, the normal Zeeman effect allow both quantum and classical treatment (by means of magnetic field effect on harmonic oscillations of the atomic electrons). One may believe that each physical analogy should have a mathematical description. For instance, optics-mechanical analogy follows from similarity between Fermat and Maupertuis (or Hamilton) principles. An analogy between Hydrodynamics and Electrodynamics is described by similarity between Helmholtz equation for the vorticity, $\mathbf{\Omega} = \nabla \times \mathbf{v}$:

$\frac{\partial \mathbf{\Omega}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{\Omega})$ and Faraday equation in the form

$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$ for the magnetic field $\mathbf{B}$.

In contrast to the examples above, the existing analogy between the ***Quantum*** and ***Classical*** Mechanics has not been yet expressed mathematically. The basic Schroedinger equation (equivalent to the Heisenberg description) is postulated (or explained) rather than being derived [1]. It is known that Erwin Schroedinger was looking for adequate mathematics which should be able to describe all the relevant experiments. He found it: this is the Schroedinger equation for the wave function Ψ, which governs the Quantum Mechanics:

$$-i\hbar \frac{\partial \Psi}{\partial t} + \hat{H}\Psi = 0\,, \tag{1}$$

where $\hat{H}$ is the Hamilton operator. However, what is its analogue in classical mechanics?

It is hard to believe that the equation he was looking for dawned upon him instantly. We will never know it for sure but let us assume that the prototype of the equation (1) was the classical Hamilton-Jacobi equation for the action S governing the classical mechanics and electrodynamics

$$\frac{\partial S}{\partial t} + H = 0\,. \tag{2}$$

There are a number of arguments in favour of this assumption. Formal similarity between equations (1) and (2) is too striking to be accidental. Ignoring the imaginary number *i*, they formally coincide with $S = \hbar\Psi$ (and hence $H = \hbar\omega$ and *vice versa*). Similarity retains if the Hamiltonian H includes the electromagnetic field [2]. Equation (1) satisfies all the requirements. It is of the first order in time, otherwise, the causality principle would not hold: the time evolution of a system is determined by the Taylor series: $\Psi(t_0 + \Delta t) = \Psi(t_0) + \left.\frac{\partial \Psi}{\partial t}\right|_{t=t_0} \cdot \Delta t$ for any moment (chosen to be initial), Δt being infinitesimal in order to ignore higher derivatives.

In other words, the equation has to include Ψ and $\frac{\partial \Psi}{\partial t}$.

It has to be linear in order to satisfy the superposition principle, necessary, for instance, to describe motion of

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atomic electrons. Of course, equation (2) is nonlinear as the classical Hamiltonian depends on quadratic term proportional to $(\nabla S)^2$. But it is easy to overcome this obstacle: replacement of the physical quantities by corresponding operators makes the equation linear as $(\nabla S)^2$ becomes $\nabla^2 S$. There is another difficulty with the equation (2): solutions of partial differential equations of the first order in time describe only irreversible processes (like diffusion or heat transfer) whereas it is necessary to consider also periodic processes like motion of atomic electrons. Therefore E Schroedinger *ad hoc* inserted the imaginary unit *i* into his equation in order to get complex solutions as well. Because of that, equation (1) cannot be derived from the known principles or equations.

In semiclassical (or WKB) approximation, when

$$\Psi = a(\mathbf{r},t)\exp(iS/\hbar)\,. \tag{3}$$

Schroedinger equation (1) is known to reduce to the classical Hamiltom-Jacobi equation (2) [1, 3] if the second-order term proportional to $\hbar^2$ is neglected. In one's turn, in the same short wavelengths limit (called also the approximation of geometrical optics) equation (2) reduces to eikonal equation of the wave optics

$$\frac{\partial\psi}{\partial t} + \omega = 0\,. \tag{4}$$

This equation sometimes is also called the Hamilton-Jacobi equation (see, e.g. [4]). Here ψ is the phase (eikonal), ω is the wave frequency. Comparing equations (2) and (4) and using the relation $H = \hbar\omega$ for the particle energy one obtains $S = \hbar\psi$. Thus, the eikonal $\psi = S/\hbar$ may be considered as the phase of the de Broglie wave Ψ (3), with the frequency $\omega = H/\hbar$ and wave vector $\mathbf{k} = m\mathbf{v}/\hbar$. We arrive at the conclusion that Hamilton-Jacobi equation (2) describes the wave-particle duality in *classical* physics. Notice that this is the only such an equation in classical physics. Of course, the roots of this duality are located in the close affinity between Fermat and Hamilton principles mentioned above.

2 On the Aharonov-Bohm effect

The Aharonov-Bohm effect was predicted in 1959 [5] in the semiclassical approximation (3). However, just in the same approximation of small wavelengths Schroedinger equation (1) reduces to the classical equations (2) and (4). This fact allows us to arrive at the same result as Aharonov and Bohm [5] without using equation (1) [6, 7].

One obtains the relation $S = \hbar\psi$ from comparison between classical equations (2) and (4) and using the relation $H = \hbar\omega$, which, by the way, has been known long before the creation of Quantum Mechanics in 1926. On the other hand, the action *S* for the system of the electric charge *q* in the magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$ is known to acquire an additional term S_{in} [8]

$$S_{in} = q\int \mathbf{A}d\mathbf{r}\,, \tag{5}$$

due to interaction between the charge *q* and the field **B**. As $\psi = S/\hbar$, the phase shift $\Delta\psi$ accumulated due to charge-field interaction equals

$$\Delta\psi = \frac{q}{\hbar}\int \mathbf{A}d\mathbf{r}\,. \tag{6}$$

Thus, we arrived at the Aharonov-Bohm effect without using the Schroedinger equation (1).

No wonder that the Aharonov-Bohm effect has classical roots. As mentioned above, normal Zeeman effect also has these roots. Its explanation by means of H. A. Lorentz electron theory holds already more than a century, along with the explanation via Quantum Mechanics. These facts may be used as additional indication of classical roots of Quantum Mechanics.

It is worth mentioning that Aharonov-Bohm effect is a source of ideas on the special role of the magnetic potential **A** in Quantum Mechanics and so called electron-solenoid unlocal interaction. E. Feinberg [9] was the first to show that the interaction between the electron current **j** = *q***v** and the solenoid, due to Faraday induction has to be taken into consideration. As a result of this interaction, the magnetic field arises outside the solenoid, and the Lorentz force $q\mathbf{v}\times\mathbf{B} \neq 0$. We believe that the very fact that the interaction term S_{in} in equation (5) results in the correct phase shift in equation (6) shows that the classical term S_{in} in the classical Hamiltonian already includes the field arising outside the solenoid due to Faraday induction. Indeed, Maxwell equations (which include the Faraday induction law) may be derived from Hamilton principle of least action *S* where the charge-field interaction is described by the term S_{in} (equation (5)) [8]. Thus, an assumption on the unlocal interaction becomes unnecessary.

As to special role of potentials (**A**, φ), Feinberg [9] underwent very thorough search but did not find any signs of their special role in experiments suggested in [5]. No wonder, because the Aharonov-Bohm effect was derived in the semiclassical approximation (3) when equation (1) reduces to classical equation (2). Then, where such a role in Quantum Mechanics arises from? The Hamilton function for the electric charge *q* in the electromagnetic field with potentials (**A**, φ) in classical electrodynamics is [8]

$$H = \frac{1}{2m}\left(\nabla S - q\mathbf{A}\right)^2 + q\varphi\,, \tag{7}$$

whereas in Quantum Mechanics it is [3]

$$\hat{H} = \frac{1}{2m}\left(\hat{P} - q\mathbf{A}\right)^2 + q\varphi\,, \tag{8}$$

where the momentum operator $\hat{P} = -i\hbar\nabla$. By using the Heisenberg uncertainty principle along with the Coulomb gauge one obtains $\hat{P}\mathbf{A} - \mathbf{A}\hat{P} = 0$, and hence the equation (8) is rewritten as

$$\hat{H} = \frac{1}{2m}\left(\hat{P}^2 - 2q\mathbf{A}\hat{P} + q^2\mathbf{A}^2\right)^2 + q\varphi\,. \tag{9}$$

Comparison between equations (7) and (9) shows that the potentials (**A**, φ) participate in Quantum Mechanics exactly in the same way as in the classical Electrodynamics, namely, as vector and scalar functions of coordinates, respectively, rather than as differential operators.

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